

# Genuine Multipartite Entanglement in Quantum Phase Transitions

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We demonstrate that the Global Entanglement (GE) measure defined by Meyer and Wallach, J. Math. Phys. **43**, 4273 (2002), is maximal at the critical point for the Ising chain in a transverse magnetic field. Our analysis is based on the equivalence of GE to the averaged linear entropy, allowing the understanding of multipartite entanglement (ME) features through a generalization of GE for bipartite blocks of qubits. Moreover, in contrast to GE, the proposed ME measure can distinguish three paradigmatic entangled states:  $GHZ_N$ ,  $W_N$ , and  $EPR^{\otimes N/2}$ . As such the generalized measure can detect genuine ME and is maximal at the critical point.

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Entanglement is a correlation of exclusively quantum nature present (in principle) in any set of post-interacting quantum systems [1]. As such multipartite entanglement (ME) is expected to play a key role on quantum phase transition (QPT) phenomena in the same way that (statistical) classical correlation does on classical phase transitions [5, 6]. In ordinary phase transitions, at the critical point, a non-zero order parameter characterizes a long range correlation (given by the correlation length divergence). In the same way, in QPTs it is expected that ME be maximal at the critical point, in the sense that all the system parties would be entangled to each other [5]. However, this conjecture could not be proved in general neither by measures of pairwise entanglement nor by the proposed ME measures. Even after a considerable effort, a deep understanding of multipartite entangled states (MES) is lacked. It is still a great challenge thus to capture the essential features of genuine ME, from a conceptual point of view, as well as from a quantitative approach, defining a measure that among other properties be able to distinguish MES [2, 3].

Indeed, concerning the legitimate quantum correlations in QPTs it would be certainly important to know exactly what kind of entanglement should we expect to be maximal at the critical point. The great majority of efforts trying to answer this question made use of two kinds of bipartite entanglement measures, both calculated for spin-1/2 lattice models such as the Ising model in a transverse magnetic field [4]. The first one, namely the pairwise entanglement (concurrence) between two spins in the chain, was studied by Refs. [5, 6]. The second one, the entropy of entanglement between one part of the chain (a block of  $L$  spins) and the rest of the chain, was investigated by Refs. [5, 7, 8]. Some candidates of ME measures were also evaluated in systems exhibiting QPTs [9, 10, 11]. Nevertheless, none of the entanglement measures employed in the above references are maximal at the critical point but the single site entropy for the Ising model [5] in the thermodynamical limit and the Localizable Entanglement [11] for an Ising chain with 14 spins. Furthermore, in Refs. [5, 6] the authors have indepen-

dently shown that bipartite entanglement vanishes when the distance between the two spins is greater than two lattice sites [12]. This is not expected since long range quantum correlations should be present at the critical point. It was then suggested that bipartite entanglement at the critical point would be decreased in order to increase ME due to entanglement sharing [5]. In other words, ME only appears at the expense of pairwise entanglement and at the critical point we should expect a genuine MES.

In this paper we demonstrate that the Global Entanglement (GE) introduced in Ref. [13] indeed captures the essential point to be maximal at the critical point for the Ising model in a transverse magnetic field in the thermodynamical limit. We also prove that there exists an interesting relation among GE, von Neumann entropy, linear entropy (LE), and 2-tangle [14, 15, 16], showing that they are all equivalent to detect QPTs. Furthermore, this relation helps us to understand the results obtained in Ref. [5], as outlined in the previous paragraph, and suggests that they are not particular to the Ising model but common to all MES with translational invariance. In addition to this, we generalize GE and propose a new ME measure, which is also maximal at the critical point for the Ising model, can detect genuine MES, and contrary to GE, furnishes different values for the entanglement of the  $GHZ_N$ ,  $W_N$ , and  $EPR^{\otimes N/2}$  states, thus being able to distinguish among MES.

For a  $N$  qubit system (spin-1/2 chain) it was noticed that GE is simply related to the  $N$  single qubit purities [16, 17, 18] by

$$E_G^{(1)} = 2 - \frac{2}{N} \sum_{j=1}^N \text{Tr}(\rho_j^2) = \frac{1}{N} \sum_{j=1}^N S_L(\rho_j) = \langle S_L \rangle, \quad (1)$$

where GE is here on identified as  $E_G^{(1)}$ ,  $\rho_j = \text{Tr}_{\bar{j}}\{\rho\}$  is the  $j$ -th qubit reduced density matrix obtained by tracing out the other  $\bar{j}$  qubits, and  $S_L(\rho_j) = \frac{d}{d-1} [1 - \text{Tr}(\rho_j^2)]$  is the standard definition of LE. This relation shows that  $E_G^{(1)}$  is just the mean of LE. It was also noticed in Refs.

[16, 19] that

$$E_G^{(1)} = \frac{1}{N} \sum_{j=1}^N \tau_{j, \text{rest}} = \langle \tau \rangle, \quad (2)$$

where  $\tau_{j, \text{rest}} = C^2$  is the 2-tangle [14, 15, 16], the square of the concurrence  $C$  [20]. Both LE and the 2-tangle can thus be used to quantify the entanglement between any block bipartition of a system of  $N$ -qubits. (They quantify the entanglement between one qubit  $j$  and the *rest*  $N - 1$  qubits of the chain [16].) The proof of (2) is based on the Schmidt decomposition [21], which also allows us to use for pure systems the reduced von Neumann entropy,  $S_V(\rho_{j(\bar{j})}) = -\text{Tr}_{j(\bar{j})} [\rho_{j(\bar{j})} \log_d(\rho_{j(\bar{j})})]$ , as a good bipartite entanglement measure [22]. Here  $d = \min\{\dim \mathcal{H}_j, \dim \mathcal{H}_{\bar{j}}\}$  and  $\dim \mathcal{H}_{j(\bar{j})}$  is the Hilbert space dimension of subsystem  $j(\bar{j})$ . Recalling that  $S_V$  is bounded from below by  $S_L$  and employing Eqs. (1) and (2) we obtain the following important relation

$$E_G^{(1)} = \langle \tau \rangle = \langle S_L \rangle \leq \langle S_V \rangle, \quad (3)$$

which states that GE is nothing but the mean LE of single qubits with the rest of the chain. Furthermore, GE is also equal to the mean 2-tangle and a lower bound for the mean von Neumann entropy. An immediate consequence of this result shows up when we deal with linear chains with translational invariance. This implies that  $\langle S_L \rangle = S_L(\rho_j)$  and that  $\langle S_V \rangle = S_V(\rho_j)$ . Hence, Eq. (3) becomes  $E_G^{(1)} = S_L(\rho_j) \leq S_V(\rho_j)$ . Since  $S_L(\rho_j)$  and  $S_V(\rho_j)$  have the same concavity and both entropies attain their maximal value for a maximally mixed state this last relation shows that  $E_G^{(1)}$  is as efficient as the linear and the von Neumann entropies to detect QPTs. In Ref. [5] the authors used  $S_V$  and in Ref. [9]  $E_G^{(1)}$  was employed to detect QPTs in the Ising model. Needless to say, both works arrived at the same results for a given range of parameters via, notwithstanding, different entanglement measures which by that time were thought to be unrelated.

Despite its success to detect the Greenberger-Horne-Zeilinger (GHZ) state [19, 23],  $E_G^{(1)}$  sometimes fails for distinguishing different multipartite states. This is best understood if we study  $E_G^{(1)}$  for three paradigmatic multipartite states. The first is  $|GHZ_N\rangle = (1/\sqrt{2})(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ , where  $|0\rangle^{\otimes N}$  and  $|1\rangle^{\otimes N}$  represent  $N$  tensor products of  $|0\rangle$  and  $|1\rangle$  respectively. The second is a tensor product of  $N/2$  Bell states [18],  $|EPR_N\rangle = |\Phi^+\rangle^{\otimes N/2}$ , where  $|\Phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ . This state is obviously not a MES. Only the pairs of qubits  $(2j-1, 2j)$ , where  $j = 1, 2, \dots, N$ , are entangled. Nevertheless, for both states  $E_G^{(1)} = 1$ . The last one is the W state [2]:  $|W_N\rangle = (1/\sqrt{N}) \sum_{j=1}^N |00 \dots 1_j \dots 00\rangle$ . The state  $|00 \dots 1_j \dots 00\rangle$  represents  $N$  qubits in which the  $j$ -th is  $|1\rangle$  and the others are  $|0\rangle$ . As shown in Ref. [13],  $E_G^{(1)}(W_N) = 4(N-1)/N^2$ .

We now present a generalization of GE. The main features of this new approach are three-fold. First, it becomes clear that we have different classes of ME measures, where  $E_G^{(1)}$  is the first one. Second, the first non trivial class,  $E_G^{(2)}$ , furnishes different values for the three states considered above. Third, it gives new insights in the study of QPT and ME.

In order to define  $E_G^{(2)}$  we need the following function

$$G(2, l) \equiv \frac{4}{3} \left( 1 - \frac{1}{N-l} \sum_{j=1}^{N-l} \text{Tr}(\rho_{j, j+l}^2) \right), \quad (4)$$

where  $\rho_{j, j+l}$  is the density matrix of qubits  $j$  and  $j+l$ , obtained by tracing out the other  $N-2$  qubits. The index  $0 < l < N$  is the distance in the chain of two qubits and  $4/3$  is a normalization constant assuring  $G(2, l) \leq 1$ . Of interest here are two quantities that can be considered ME measures in the same sense that  $E_G^{(1)}$  is:

$$G(2, 1) \equiv \frac{4}{3} \left( 1 - \frac{1}{N-1} \sum_{j=1}^{N-1} \text{Tr}(\rho_{j, j+1}^2) \right), \quad (5)$$

and

$$E_G^{(2)} = \frac{1}{N-1} \sum_{l=1}^{N-1} G(2, l). \quad (6)$$

We can interpret  $G(2, 1)$  as the mean LE of all two qubit nearest neighbors with the rest of the chain. Similar interpretations are valid for the others  $G(2, l)$ .  $E_G^{(2)}$  is the mean of all  $G(2, l)$  and it gives the mean LE of all two qubits, independent of their distance, with the rest of the chain [24]. To define  $E_G^{(3)}$  we need the function  $G(3, l_1, l_2)$  with one more parameter, since now we can have different distances among the three qubits of the reduced state. A complete analysis of this new ME measure and its usefulness to detect MES is discussed elsewhere [25].

Table I shows the quantities given by Eqs. (5) and (6) for  $GHZ_N$ ,  $EPR_N$ , and  $W_N$ . We note that due to translational symmetry,  $G(2, 1)$  and  $E_G^{(2)}$  are identical for  $GHZ_N$  and  $W_N$ . It is worthy of mention that depending on the value of  $N$ , the states are differently classified by  $G(2, 1)$ . A similar behavior is observed for  $E_G^{(2)}$  [24]. In this case, however,  $EPR_N$  is the most entangled state for long chains. The reason for that lies on the definition of  $E_G^{(2)}$ . For  $EPR_N$ ,  $G(2, l) = 1$  for any  $l \geq 2$ . Thus, since  $E_G^{(2)}$  is the average of all  $G(2, l)$ , for long chains  $G(2, 1)$  does not contribute significantly and  $E_G^{(2)} \rightarrow 1$ .

It is worth noticing that even at the thermodynamical limit,  $N \rightarrow \infty$ ,  $E_G^{(2)}$  and  $G(2, 1)$  still distinguish the three states. However, the ordering of the states is different. As already explained, this is due to the contribution of  $G(2, l)$ ,  $l \geq 2$ , in the calculation of  $E_G^{(2)}(EPR_N)$ .

Now we specify to the one-dimensional Ising model in a transverse magnetic field, which is given by the following

Table I: Comparison among the three paradigmatic states.

	$E_G^{(1)}$	$G(2,1)$	$E_G^{(2)}$
$GHZ_N$	1	2/3	2/3
$EPR_N$	1	$\frac{N-2}{2(N-1)}$	$\frac{(2N-1)(N-2)}{2(N-1)^2}$
$W_N$	$\frac{4(N-1)}{N^2}$	$\frac{16(N-2)}{3N^2}$	$\frac{16(N-2)}{3N^2}$

Hamiltonian

$$H = \lambda \sum_{i=1}^N \sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^N \sigma_i^z, \quad (7)$$

where  $i$  represents the  $i$ -th qubit,  $\lambda$  is a free parameter related to the inverse strength of the magnetic field, and we work in the thermodynamical limit. We assume periodic boundary conditions:  $\sigma_{N+1} = \sigma_1$ . As we have shown, for a system with translational symmetry GE is nothing but LE of one spin with the rest of the chain. We only need, then, LE to obtain GE. For that end we must calculate the single qubit (or single site) reduced density matrix, which is obtained from the two qubits (two sites) reduced density matrix. It is a  $4 \times 4$  matrix and can be written as

$$\rho_{ij} = \text{Tr}_{ij}[\rho] = \frac{1}{4} \sum_{\alpha, \beta} p_{\alpha\beta} \sigma_i^\alpha \otimes \sigma_j^\beta, \quad (8)$$

where  $\rho$  is the broken-symmetry ground state in the thermodynamical limit and  $p_{\alpha\beta} = \text{Tr}[\sigma_i^\alpha \sigma_j^\beta \rho_{ij}] = \langle \sigma_i^\alpha \sigma_j^\beta \rangle$ .  $\text{Tr}_{ij}$  is the partial trace over all degrees of freedom except the spins at sites  $i$  and  $j$ ,  $\sigma_i^\alpha$  is the Pauli matrix acting on the site  $i$ ,  $\alpha, \beta = 0, x, y, z$  where  $\sigma^0$  is the identity matrix, and  $p_{\alpha\beta}$  is real. Therefore, all we need are the ground state two-point correlation functions (CFs). By symmetry arguments concerning the ground state [5] the only non-zero CFs are  $p_{00}, p_{xx}, p_{yy}, p_{zz}, p_{0x} = p_{x0}, p_{0z} = p_{z0}$ , and  $p_{xz} = p_{zx}$ . Due to normalization  $p_{00} = 1$  and a direct calculation gives  $p_{xz} = p_{zx} = 0$  for  $\lambda \leq 1$ . On the other hand, the Schwartz inequality necessarily gives  $0 \leq |p_{xz}| \leq |\langle \sigma_i^x \rangle \langle \sigma_i^z \rangle|$ , allowing thus that the lower and upper bounds for entanglement be calculated for  $\lambda > 1$ . We plot the upper bound for entanglement by taking  $p_{xz} = 0$ . By continuity the true value for entanglement must show a similar behavior.

Those CFs have been already calculated [4] and we just highlight the main results. The two-point CFs and the mean values of  $\sigma^x$  and  $\sigma^z$  are

$$\langle \sigma_1^x \sigma_l^x \rangle = \begin{vmatrix} g(-1) & g(-2) & \cdots & g(-l) \\ g(0) & g(-1) & \cdots & g(-l+1) \\ \vdots & \vdots & \ddots & \vdots \\ g(l-2) & g(l-3) & \cdots & g(-1) \end{vmatrix}, \quad (9)$$

$$\langle \sigma_1^y \sigma_l^y \rangle = \begin{vmatrix} g(1) & g(0) & \cdots & g(-l+2) \\ g(2) & g(1) & \cdots & g(-l+3) \\ \vdots & \vdots & \ddots & \vdots \\ g(l) & g(l-1) & \cdots & g(1) \end{vmatrix}, \quad (10)$$

$\langle \sigma_1^z \sigma_l^z \rangle = \langle \sigma_1^z \rangle^2 - g(l)g(-l)$ ,  $\langle \sigma_1^z \rangle = g(0)$ , and  $\langle \sigma_1^x \rangle = 0$  for  $\lambda \leq 1$  or  $\langle \sigma_1^x \rangle = (1 - \lambda^{-2})^{1/8}$  for  $\lambda > 1$ . Here  $g(l) = \mathcal{L}(l) + \lambda \mathcal{L}(l+1)$ ,  $\mathcal{L}(l) = \frac{1}{\pi} \int_0^\pi dk \frac{\cos(kl)}{1 + \lambda^2 + 2\lambda \cos(k)}$ , and  $l \geq 1$  is the lattice site distance between two qubits. By tracing out one of the qubits we obtain the single qubit density matrix, which allows us to obtain  $E_G^{(1)}$  as a function of  $\lambda$ . This is shown in Fig. 1. As a matter of fact  $E_G^{(1)}$  is maximal (with singular derivative) at the critical point  $\lambda = 1$ . For comparison, in Fig. 1 we plot  $S_V(\rho_j)$ , which was already shown also maximal at the critical point for the broken-symmetry state [5]. We em-

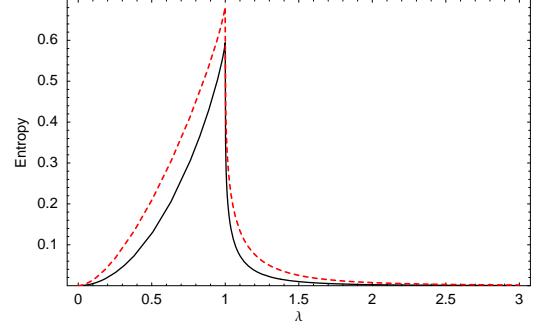


Figure 1: (Color online) Von Neumann entropy (dashed) and GE/LE (solid) as a function of  $\lambda$ .

phasize that these measures quantify entanglement in the global system by measuring how *mixed* the subsystems are. The physical meaning behind studying “mixedness” lies on the fact that the more entangled two subsystems are the more mixed their reduced density matrix should be [9, 18]. However, in a many-body system there are many ways in which one could divide the global system into subsystems. The first non-trivial generalization is to study LE of two sites with the rest of the chain. Using  $\rho_{ij}$  we can calculate  $G(2, l)$  for the Ising model (Fig. 2). It has a similar behavior to  $E_G^{(1)}$ , being also maximal (with singular derivative) at the critical point. This feature demonstrates that both a pair of nearest neighbors sites and the sites themselves are maximally entangled to the rest of the chain at the critical point. But this is not particular to nearest neighbors as shown in Fig. 3, where  $G(2, 1)$ ,  $G(2, 15)$ , and  $E_G^{(2)} = \frac{1}{15} \sum_{i=1}^{15} G(2, i)$  is plotted.  $G(2, 15)$  is also maximal at the critical point, indicating that in a QPT entanglement sharing at the critical point is favored by an increase of all kind of ME. Moreover, Fig. 3 shows that  $G(2, 15)$  is only slightly different from  $E_G^{(2)} = \frac{1}{15} \sum_{i=1}^{15} G(2, i)$ . This is due to the rapid convergence of  $G(2, l)$  as  $l$  is increased. At the critical point  $\lim_{l \rightarrow \infty} G(2, l)$  is 0.675, and thus higher than the value for  $GHZ_N$ ,  $EPR_N$ , and  $W_N$ , obtained in the thermodynamical limit, indicating thus a genuine MES. We also remark that besides  $E_G^{(1)}$ ,  $G(2, l)$ , and  $E_G^{(2)}$  being all maximal at the critical point,  $E_G^{(1)} < E_G^{(2)}$  for every value

of  $\lambda$ . However an interesting change of ordering for  $E_G^{(1)}$  and  $G(2,1)$  occurs around the critical point. For  $\lambda \leq 1$ ,  $E_G^{(1)} > G(2,1)$ , but for  $\lambda > 1$ ,  $E_G^{(1)} < G(2,1)$ . Thence a kind of ME is favored in detriment of the other, depending on the system phase. Also, the fact that at the critical point both  $E_G^{(1)}$  and  $E_G^{(2)}$  are maximal indicates entanglement sharing, such that all the sites of the chain are strongly (quantum) correlated. Of course this statement is only completely true provided that  $E_G^{(m)}$  is also shown to be maximal for any  $2 < m \leq N-1$  (all possible partitions). Furthermore, the fact that  $G(2,l)$  always increase as  $l \rightarrow \infty$  at the critical point suggests a kind of diverging entanglement length. However its precise definition demands a careful calculation of the scaling of entanglement such as in Refs. [7, 9]. These points are left for further investigation [25].

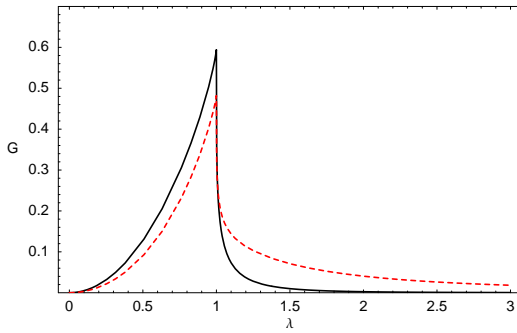


Figure 2: (Color online)  $E_G^{(1)}$  (solid) and  $G(2,1)$  (dashed) as a function of  $\lambda$ . Both quantities are maximal at the critical point  $\lambda = 1$ .

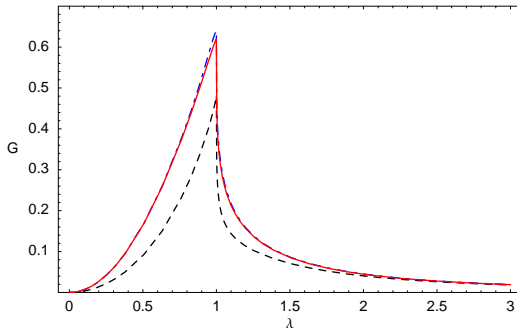


Figure 3: (Color online)  $G(2,1)$  (dashed/black),  $G(2,15)$  (solid/red), and  $E_G^{(2)}$  (dotted-dashed/blue) as a function of  $\lambda$ . We see that  $E_G^{(2)}$  is slightly different from  $G(2,15)$ , showing that  $G(2,l)$  saturates as  $l \rightarrow \infty$ .

In conclusion we have demonstrated that for an infinite Ising chain both  $E_G^{(1)}$  and its generalization,  $E_G^{(2)}$ , are maximal at the critical point. Furthermore,  $E_G^{(2)}$  as defined here is able to detect genuine ME. We remark that the behavior of the ME measures here presented for an infinite chain is in agreement with the Localizable En-

tanglement calculated for a finite ( $N=14$ ) Ising chain for the broken-symmetry state [11]. Yet our results were obtained in a relatively simpler fashion and could be used to infer genuine ME for systems where the Localizable Entanglement has failed to detect QPT [26]. Finally, our results reinforced Osborne and Nielsen [5] conjecture that at the critical point ME should be high, due to entanglement sharing, in detriment of bipartite entanglement.

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